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Critical levels in a jet-type flow

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The reflection and transmission of a gravity wave propagating through a jet-type background flow is studied. Only the linear, non-dissipative case is treated, and the hydrostatic approximation used in a stratified non-rotating medium. The behaviour of the gravity wave in the presence of two or one critical levels is investigated. In the first case, i.e. two critical levels, it is found that for high values of the Richardson number the wave is highly attenuated. For sufficiently low values of the Richardson number overreflection and overtransmission occur. It is demonstrated that a wave generated below the jet and propagating upward takes energy from the mean flow at the upper critical level for all values of the Richardson number. The single critical level has been studied as a limiting case of two merging critical levels. In this approach it is found that the wave is not transmitted and no overreflection can occur.

1. Introduction

In a classical paper, Booker & Bretherton (1967) analysed the propagation of an internal gravity wave through a height-dependent wind field containing one critical level, i.e. a level where the wind velocity equals the horizontal phase velocity of the wave. They showed that the transmission of the wave depends only on the value of the Richardson number at the critical level. They considered only background flow with Richardson number larger than 0.25. Later on, Jones (1968) found that, for low values of the Richardson number at the critical level, overreflection, i.e. a reflection coefficient larger than 1, occurs.

These results have been confirmed by various authors and for different background flows, e.g. a broken-line profile (Eltayeb & McKenzie 1975) and a hyperbolic-tangent profile (van Duin & Kelder 1982).

Viscosity and thermal conduction were introduced by Hazel (1967). He showed that for values of the Richardson number larger than 0.25 a large amount of wave energy is lost near the critical level. The transmission coefficient is the same as found by Booker & Bretherton in the dissipationless model.

Geller, Tanaka & Fritts (1975) and Fritts & Geller (1976) studied the instability in the vicinity of the critical level. They found that viscosity and heat conduction can have a strongly stabilizing role. A numerical model was used by Fritts (1978, 1982) to compare the effects of viscosity, time dependence and nonlinear interaction. Time dependence is found to play only a minor role in stabilizing the critical level. Nonlinear effects can give rise to higher harmonics of the forcing wave, which develop large amplitudes near the critical level when viscous effects are small.

A nonlinear non-dissipative treatment was given by Brown & Stewartson (1980, 1982*a*, *b*). They showed that for large values of the Richardson number the linear model is valid up to a certain time inversely proportional to the wave amplitude. After that time, the reflection and transmission coefficients change. Another very different approach to the nonlinear stationary problem was given by Teitelbaum & Sidi (1979). They showed that a contact discontinuity appears below the critical level in the absence of dissipative phenomena.

To summarize: nonlinearities might change the results of linear models to an extent that is not well understood either theoretically or experimentally. In agreement with Lindzen (1973), we believe that, if dissipative damping occurs before amplitudes have grown to the point where nonlinear effects become important, the linear approximation is a good one.

The problem of one critical level has been extensively treated by Rosenthal & Lindzen (1983*a*, *b*) and Lindzen & Rosenthal (1983) with regard to instabilities and the relation between instabilities and overreflection. Grimshaw (1980) included rotation and electrical conductibility in the linear and inviscid problem of one critical level. Although he refers to the problem of two or more critical levels, he does not study it.

Propagation through background flows containing more than one critical level has attracted little interest. Drazin, Zaturska & Banks (1979) have done a calculation for a flow containing two critical levels. They modelled the flow by a broken-line profile. They showed that for large values of the Richardson number the transmission coefficient equals that of two independent critical levels having the same value of the Richardson number.

In this paper propagation through two critical levels in a jet-type background flow is studied analytically and numerically.

As the mathematical treatment is more difficult than in the case of one critical level, we treat only the linear non-dissipative case.

First, we have taken a symmetric jet-type background flow. This case can be solved analytically. The reflection and transmission coefficients are determined. The influence of the distances between the two critical levels is considered. Also, various values of the Richardson number are taken. The reflection and transmission coefficients are also calculated using a numerical approach, which gives the same results as the analytical one.

The case of an asymmetric flow is also considered in the numerical approach.

Finally the limit of two critical levels approaching each other is considered in order to model the case of a single critical level.

2. The governing equations

We consider a compressible and stratified fluid. Viscosity and thermal conduction are assumed to be negligible. The horizontal background flow is only height-dependent. Perturbations to the zeroth-order state are assumed to be small to allow the linearization of the equations of conservation of mass, momentum and entropy. On this set of differential equations the hydrostatic approximation is applied. This assumption is valid if the vertical scale of motions is much smaller than the horizontal scale (Orlanski 1981; Gill 1982). As we restrict this study to gravity waves with periods long compared with the Brunt–Väisälä period, the hydrostatic approximation can be used. Furthermore, owing to the behaviour of waves near the critical level, where the ratio of the vertical to the horizontal scale tends to zero, the hydrostatic approximation becomes more and more justified.

We start with the linearized equations (cf. Holton 1975):

$$\frac{\partial u}{\partial t}+\overline{U}\frac{\partial u}{\partial x}+\frac{\partial \overline{U}}{\partial z}w+\frac{\partial \phi}{\partial x}=0, \tag{2.1}$$

$$\frac{\partial^2 \phi}{\partial t\,\partial z}+\overline{U}\frac{\partial^2 \phi}{\partial x\,\partial z}+N^2 w=0, \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{1}{p}\frac{\partial}{\partial z}(pw)=0, \tag{2.3}$$

where $z = -H\ln(p/p_0)$, H is the scale height (assumed to be constant), $\overline{U} = \overline{U}(z)$ is the background flow, w is the vertical velocity perturbation, u is the horizontal velocity perturbation, ϕ is the geopotential perturbation, N is the Brunt–Väisälä frequency, p is the pressure perturbation and p_0 a reference pressure level.

We look for normal-mode solutions, i.e. solutions with x- and t-dependence of the form $\exp[\mathrm{i}(\sigma t-kx)]$, where σ is the frequency and k the horizontal wavenumber of the wave.

If the set of equations (2.1)–(2.3) is solved for the z-dependence of ϕ, the following equation is obtained:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}-\frac{1}{H}\frac{\mathrm{d}\phi}{\mathrm{d}z}+\frac{N^2k^2}{\Omega^2}\phi=0, \tag{2.4}$$

where $\Omega = \sigma - k\overline{U}$ is the Doppler-shifted frequency.

Inserting $A(z) = \mathrm{e}^{-z/2H}\,\phi(z)$ into (2.4) leads to

$$\frac{\mathrm{d}^2A}{\mathrm{d}z^2}+\left[\frac{N^2k^2}{\Omega^2}-\frac{1}{4H^2}\right]A=0. \tag{2.5}$$

Equation (2.5) contains no derivatives of the background flow. Hence, applying the hydrostatic approximation and using as vertical coordinate log pressure gives for the geopotential perturbation a quite simple equation.

3. The solutions

The background flow $\overline{U}(z)$ is taken to be of the form

$$\overline{U}(z)=\frac{U_0}{1+z^2/D^2}. \tag{3.1}$$

This profile represents a symmetric jet-type flow.

If the horizontal phase velocity of the wave is smaller than U_0 the wave encounters two critical levels in the flow. This case will be examined first.

After introducing the independent variable $y = z/Dd$, where $d = (U_0/c-1)^{\frac{1}{2}}$ and $c = \sigma/k$ is the horizontal phase velocity of the wave, (2.5) becomes

$$\frac{\mathrm{d}^2A}{\mathrm{d}y^2}+\left(\frac{N^2D^2(1+d^2y^2)^2}{c^2d^2(y^2-1)^2}-\frac{D^2d^2}{4H^2}\right)A=0. \tag{3.2}$$

Furthermore, if we define the function $B(y) = (y^2-1)^{-\frac{1}{2}}A(y)$, the equation for B is

$$(1-y^2)\frac{\mathrm{d}^2B}{\mathrm{d}y^2}-2y\frac{\mathrm{d}B}{\mathrm{d}y}+\left(\lambda+\gamma^2(1-y^2)-\frac{\mu^2}{1-y^2}\right)B=0, \tag{3.3}$$

$$\left.\begin{aligned} &\text{where} && \lambda = -2\frac{D^2N^2U_0}{c^3}, \quad \gamma^2 = D^2d^2\left(\frac{N^2}{c^2}-\frac{1}{4H^2}\right), \\ &\text{and} && \mu^2 = 1-4Ri_c, \quad \text{with} Ri_c = \frac{D^2N^2U_0^2}{4c^4d^2} \end{aligned}\right\} \tag{3.4}$$

The Richardson number Ri is defined as $Ri \equiv N^2/(\mathrm{d}\overline{U}/\mathrm{d}z)^2$. It is easy to verify that Ri_c is the Richardson number at the critical level.

Equation (3.3) is known as the differential equation of spheroidal wave functions. Its properties and solutions can be found in Meixner & Schäfke (1954) (hereinafter referred to as MS) and Erdélyi *et al.* (1953).

The equation has three singular points at $y = \pm 1$ and ∞. The points ± 1 are regular, whereas the one at ∞ is an irregular singularity. The parameter μ is called the order of the wave function.

The spheroidal wave equation (3.3) has different solutions. As we are considering a reflection and transmission problem, we need solutions that are asymptotically approximated by plane waves.

The solutions with these properties are $S_\nu^{\mu(3,4)}(y,\gamma)$. They can be represented by convergent series for $|y| > 1$ by

$$S_\nu^{\mu(3,4)}(y,\gamma) = \frac{(y^2-1)^{-\frac{1}{2}\mu}\,y^\mu}{A_\nu^\mu(\gamma^2)} \sum_{r=-\infty}^{\infty} a_{\nu,2r}^\mu(\gamma^2)\,\Psi_{\nu+2r}^{(3,4)}(\gamma y), \tag{3.5}$$

where $A_\nu^\mu(\gamma^2) = \Sigma_{r=-\infty}^{\infty}(-1)^n a_{\nu,2r}^\mu(\gamma^2)$ and $\Psi_{\nu+2r}^{(j)}$ are the spherical Hankel functions. The parameter ν is called the characteristic exponent of the spheroidal differential equation, and is a function of λ, μ and γ^2.

It is possible to expand λ in a power series in γ^2, with coefficients depending on μ and ν (MS) as

$$\lambda_\nu^\mu(\gamma^2) = \nu(\nu+1) - \frac{1}{2}\left[1+\frac{(2\mu-1)(2\mu+1)}{(2\nu-1)(2\nu+3)}\right]\gamma^2 + \dots \tag{3.6}$$

It is obvious, and will be useful below, that

$$\lambda_\nu^\mu(0) = \nu(\nu+1). \tag{3.7}$$

Properties of these solutions and the following can be found in MS.

Other solutions we need are $Ps_\nu^\mu(y,\gamma^2)$ and $Qs_\nu^\mu(y,\gamma^2)$. They may be represented in the following forms:

$$\left.\begin{aligned} Ps_\nu^\mu(y,\gamma^2) &= \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^\mu(\gamma^2)\,P_{\nu+2r}^\mu(y), \\ Qs_\nu^\mu(y,\gamma^2) &= \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^\mu(\gamma^2)\,Q_{\nu+2r}^\mu(y), \end{aligned}\right\} \tag{3.8}$$

where P_ν^μ and Q_ν^μ are the Legendre functions of the first and second kind respectively. The series in (3.8) converge everywhere, with the possible exceptions ± 1 and ∞.

The asymptotic behaviour of $S_\nu^{\mu(3)}(y,\gamma)$ as $y \to \infty$ is

$$S_\nu^{\mu(3)}(y,\gamma) \approx (y^2-1)^{-\frac{1}{2}\mu}\,y^\mu\,\frac{\exp\{\mathrm{i}[\gamma y - \frac{1}{2}(\nu+1)\pi]\}}{\gamma y}\left\{\sum_{s=0}^{q-1}\frac{A_{\nu,s}^\mu}{[-2\mathrm{i}\gamma y]^s} + O(y^{-q})\right\}, \tag{3.9}$$

for $\pi + \epsilon \leqslant \arg(\gamma y) \leqslant 2\pi - \epsilon$, $\epsilon > 0$.

For $S_\nu^{\mu(4)}(y,\gamma)$ the asymptotic form is

$$S_\nu^{\mu(4)}(y,\gamma) \approx (y^2-1)^{-\frac{1}{2}\mu}\, y^\mu \frac{\exp\{-\mathrm{i}[\gamma y - \frac{1}{2}(\nu+1)\pi]\}}{\gamma y}\left\{\sum_{s=0}^{q-1}\frac{A^\mu_{\nu,s}}{[2\gamma y]^s}+O(y^{-q})\right\}, \quad (3.10)$$

for $-2\pi+\epsilon \leqslant \arg(\gamma y) \leqslant \pi-\epsilon,\ \epsilon > 0$.

From (3.9) and (3.10) it follows that

$$A(z) \approx \frac{\exp\{\pm \mathrm{i}[lz-\frac{1}{2}(\nu+1)\pi]\}}{\gamma} \quad \text{as } z\rightarrow\infty \quad (3.11)$$

for $|\arg(lz)| < \pi$, where $l = [N^2/c^2-1/4H^2]^{\frac{1}{2}}$ is the vertical wavenumber without mean flow.

Since $A(z) = \mathrm{e}^{-z/2H}\phi(z)$, ϕ tends exponentially to infinity as $z\rightarrow\infty$. This is a consequence of density stratification. Nevertheless, it must be taken into account that the wavelike form (3.11) is a good solution where the mean-flow velocity becomes negligible with respect to the horizontal phase velocity of the wave. Thus, in the problem at hand, the mathematical limit $z\rightarrow\infty$ means $\overline{U}\rightarrow 0$.

The plus and minus signs in (3.11) correspond to the solutions $S_\nu^{\mu(3)}$ and $S_\nu^{\mu(4)}$ respectively.

The vertical wave-energy flux, at least when the mean background flow is zero, may be written as

$$F_\mathrm{w} = \rho_0\overline{\phi w} = \tfrac{1}{2}\rho_0 \,\mathrm{Re}\,(\phi w^*), \quad (3.12)$$

where the overbar refers to the average over one cycle of the wave and ρ_0 is the basic-state density.

Equation (2.2), with the temporal and horizontal dependence used here, gives

$$w = -\mathrm{i}\frac{\Omega}{N^2}\phi_z. \quad (3.13)$$

With (3.13), the vertical wave energy flux (3.10) becomes

$$F_\mathrm{w} = \mathrm{Re}\left[\frac{\mathrm{i}\Omega}{2N^2}\rho_0\phi\phi_z^*\right], \quad (3.14)$$

and, with the asymptotic form (3.9),

$$F_\mathrm{w} = \pm\rho_0\frac{\Omega l}{2N^2}, \quad (3.15)$$

where the plus and minus signs correspond to the plus and minus signs in (3.11). Thus in the upper half-space, $S_\nu^{\mu(3)}$ corresponds to an upward-, and $S_\nu^{\mu(4)}$ to a downward-, propagating wave.

4. The reflection and transmission coefficients

Suppose there is a source emitting waves at $-\infty$. For $z\rightarrow+\infty$ we must only have an upward-propagating wave. This boundary condition may be fulfilled as seen at the end of §3 by $S_\nu^{\mu(3)}(y,\gamma)$. It is then necessary to find the analytical continuation for values of $z\rightarrow-\infty$. The physically meaningful path of analytical continuation is found by adding a small dissipation term to (2.1) and (2.2) (Booker & Bretherton 1967; Baldwin & Roberts 1970).

We can parametrize a small dissipation in terms of linear friction (Rayleigh

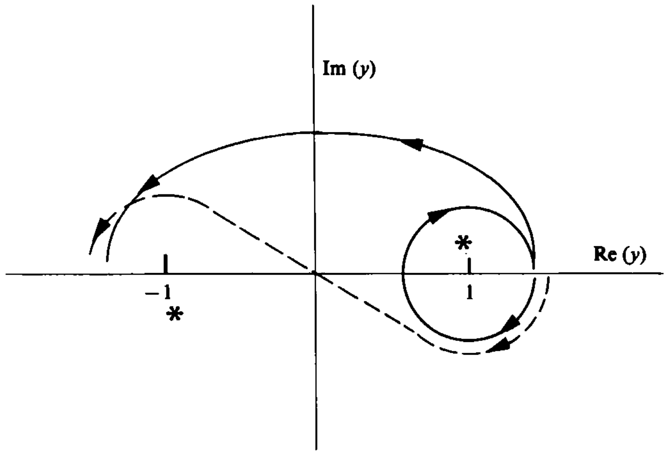

FIGURE 1. Complex y-plane showing the two equivalent paths for analytical continuation of the spheroidal wave function: ---, direct path; ——, indirect path used in our calculation; *, singularities shifted as a consequence of the addition of a small dissipation.

friction) and linear thermal damping (Newtonian cooling). This leads to a complex horizontal phase velocity $c-\mathrm{i}\epsilon$. If terms of higher order than ϵ are neglected, the singularities in (3.3) are at $y = \pm\alpha$, where

$$\alpha = 1+\mathrm{i}\epsilon', \quad \text{with } \epsilon' = \frac{U_0}{2c^2d^2}\epsilon \ll 1.$$

A Frobenius expansion near $y = \alpha$ starts with the terms

$$[y-(1+\mathrm{i}\epsilon')]^{\pm\frac{1}{2}\mu'},$$

where

$$\mu'^2 = 1-4\,Ri_c\left(1+\frac{2\mathrm{i}\epsilon}{c}\right).$$

If $\epsilon = 0$ there is a branch point at $y = 1$. If $\epsilon > 0$, as is the case when we consider small dissipation, then, as $y-1$ decreases from positive values that are large compared with ϵ' to negative values, the argument of $y-1-\mathrm{i}\epsilon'$ changes continuously from 0 to $-\pi$.

Near $y = -\alpha$ the first term of a Frobenius expansion is $[y+(1+\mathrm{i}\epsilon')]^{\pm\frac{1}{2}\mu'}$. In this case, as $y+1$ decreases from positive to negative values, the argument of $y+1+\mathrm{i}\epsilon'$ changes continuously from 0 to π.

The above analysis shows that the connection path can be one of the equivalent paths shown in figure 1. We followed the path indicated by a solid line, allowing us to use known properties of the solutions, as we shall see later.

The $S_\nu^{\mu(3,4)}$ are only defined for $|y| > 1$. First we have to extend the functions into the unit circle. This may be done with the following relations:

$$\left.\begin{aligned} S_\nu^{\mu(3)}(y,\gamma) &= \frac{1}{\cos\nu\pi}\,[-\mathrm{i}\,V^\mu_{-\nu-1}\,\widetilde{Qs}{}_\nu^\mu(y,\gamma^2)+\mathrm{e}^{-\mathrm{i}\nu\pi}\,V^\mu_\nu\,\widetilde{Qs}{}^\mu_{-\nu-1}(y,\gamma^2)],\\ S_\nu^{\mu(4)}(y,\gamma) &= \frac{1}{\cos\nu\pi}\,[\mathrm{e}^{\mathrm{i}\nu\pi}\,V^\mu_\nu\,\widetilde{Qs}{}^\mu_{-\nu-1}(y,\gamma^2)+\mathrm{i}V^\mu_{-\nu-1}\,\widetilde{Qs}{}^\mu_\nu(y,\gamma)],\end{aligned}\right\} \quad (4.1)$$

where

$$\widetilde{Qs}{}_\nu^\mu(y,\gamma^2) = \frac{\mathrm{e}^{-\mathrm{i}\mu\pi}}{\Gamma(\nu+\mu+1)}\,Qs_\nu^\mu(y,\gamma^2)$$

and V_ν^μ is a constant, which for small values of γ^2 can be approximated by

$$V_\nu^\mu(\gamma^2) \approx \frac{1}{2}\left(\frac{\gamma}{4}\right)^\nu \frac{\Gamma(-\nu+\frac{1}{2})}{\Gamma(\nu+\frac{3}{2})}(1-O(\gamma^2)). \tag{4.2}$$

The connection path can be split into two parts. First we turn around $y = 1$ through -2π. In MS it is proved that

$$Ps_\nu^\mu[1+(y-1)\,e^{-2i\pi}] = e^{\mu\pi i}\,Ps_\nu^\mu(y); \tag{4.3}$$

accordingly

$$\widetilde{Qs}_\nu^\mu[1+(y-1)\,e^{-2\pi i}] = e^{i\nu\pi}\frac{\cos\mu\pi}{\cos\nu\pi}\widetilde{Qs}_\nu^\mu(y) - i\frac{\sin(\nu-\mu)\pi\,\Gamma(\mu-\nu)}{\cos\nu\pi\,\Gamma(\nu+\mu+1)}\widetilde{Qs}_{-\nu-1}^\mu(y). \tag{4.4}$$

Next we turn around $y = 0$ through $+\pi$, which gives

$$\widetilde{Qs}_\nu^\mu(y\,e^{i\pi}) = e^{-i(\nu+1)\pi}\,\widetilde{Qs}_\nu^\mu(y). \tag{4.5}$$

Inserting (4.5) into (4.4) gives

$$\widetilde{Qs}_\nu^\mu(-y) = -\frac{\cos\mu\pi}{\cos\nu\pi}\widetilde{Qs}_\nu^\mu(y) - i\frac{\sin(\nu-\mu)\pi\,\Gamma(\mu-\nu)}{\cos\nu\pi\,\Gamma(\nu+\mu+1)}e^{i\nu\pi}\,\widetilde{Qs}_{-\nu-1}^\mu(y). \tag{4.6}$$

In the same way, an expression for $Qs_{-\nu-1}^\mu(-y)$ may be derived.

With the help of (4.1), we can go back to $S_\nu^{\mu(3,4)}$, and the result is

$$S_\nu^{\mu(3)}(-y,\gamma) = pS_\nu^{\mu(3)}(y,\gamma) + qS_\nu^{\mu(4)}(y,\gamma), \tag{4.7}$$

with
$$p = \frac{2\cos\mu\pi\,\sin\nu\pi + i[V^+\,e^{-i\nu\pi} - V^-\,e^{i\nu\pi}]}{2i\cos^2\nu\pi} \tag{4.8}$$

and
$$q = \frac{i\,e^{-i\nu\pi}[2\cos\nu\pi - V^+\,e^{-2i\nu\pi} - V^-\,e^{2i\nu\pi}]}{2i\cos^2\nu\pi}, \tag{4.9}$$

where
$$V^+ = \sin(\nu+\mu)\,\pi\frac{\Gamma(\mu+\nu+1)\,V_\nu^\mu(\gamma^2)}{\Gamma(\mu-\nu)\,V_{-\nu-1}^\mu(\gamma^2)} \tag{4.10}$$

and
$$V^- = \sin(\nu-\mu)\,\pi\frac{\Gamma(\mu-\nu)\,V_{-\nu-1}^\mu(\gamma^2)}{\Gamma(\mu+\nu+1)\,V_\nu^\mu(\gamma^2)}. \tag{4.11}$$

If we let $y \to \infty$ in (4.7) we obtain, with the asymptotic forms (3.9) and (3.10), a relation between plane waves.

If we take into account that for negative y, $S_\nu^{\mu(4)}$ represents the incident wave and $S_\nu^{\mu(3)}$ the reflected one, the coefficients of reflection R and transmission T become

$$R = \frac{p}{e^{i\nu\pi}\,q} \tag{4.12}$$

and
$$T = \frac{1}{e^{i\nu\pi}\,q}, \tag{4.13}$$

with p and q defined above.

The parameters are μ, γ and λ, but the number of parameters may be reduced if we consider that the horizontal phase velocity of the wave c is much smaller than the sound velocity. In the atmosphere the sound velocity is $V_s \approx 4N^2H^2$, and with

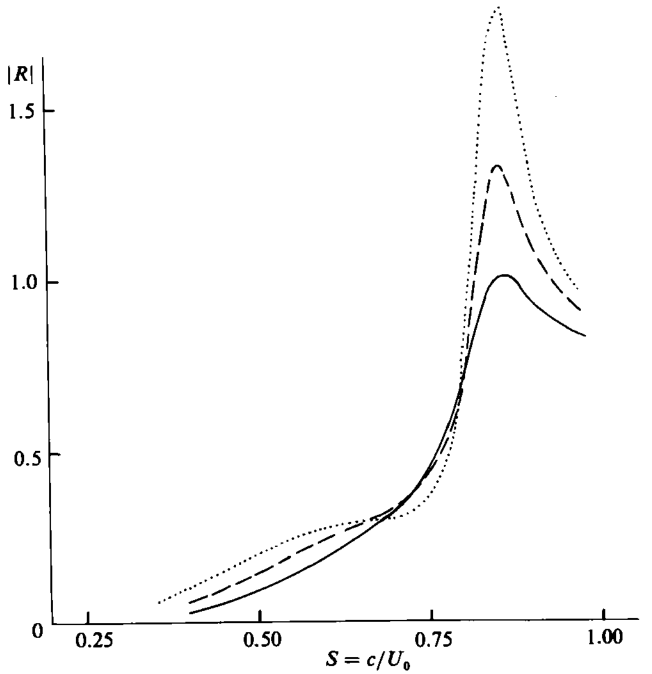

FIGURE 2. Variation of the reflection coefficient $|R|$ as a function of $S = c/U_0$, for three different values of the minimum Richardson number of the mean flow: ——, $Ri_m = 0.143$; ---, 0.1; ..., 0.07.

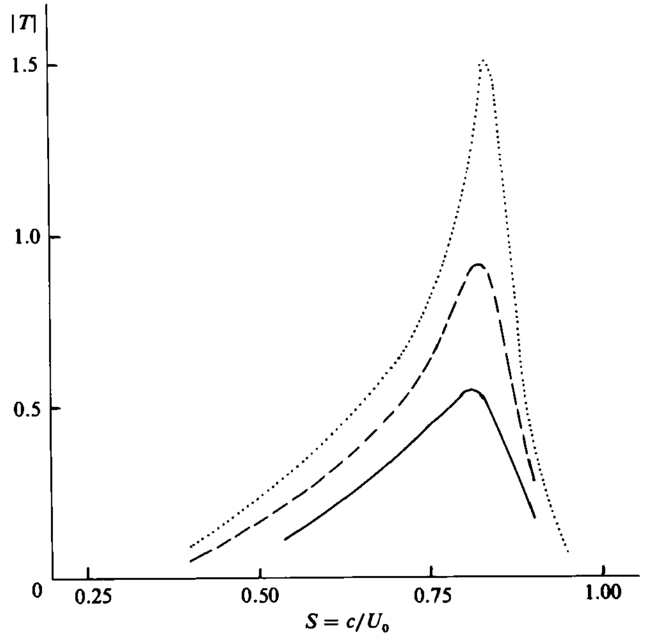

FIGURE 3. Variation of the transmission coefficient $|T|$ as a function of $S = c/U_0$, for three different values of the minimum Richardson number of the mean flow: ——, $Ri_m = 0.143$; ---, 0.1; ..., 0.07.

this approximation it is possible to write

$$\left.\begin{aligned} \lambda = \frac{2J}{S^3}, \quad \mu^2 = 1-\frac{J}{S^3(1-S)}, \quad \gamma^2 \approx \frac{J(1-S)}{S^3}, \\ \text{where} \qquad J = \frac{N^2D^2}{U_0^2}, \quad S = \frac{c}{U_0}. \end{aligned}\right\} \tag{4.14}$$

Note that this approximation, which can be interpreted as the incompressible-fluid approximation ($H \rightarrow \infty$), has not been used elsewhere in this study. It is only adopted to simplify the presentation and interpretation.

Thus only two parameters are involved: J and the ratio S between the phase velocity and the maximum velocity of the background flow.

Another useful parameter is the minimum Richardson number of the flow Ri_{m}, which corresponds to $S = 0.75$ and is equal to $(\frac{4}{3})^3 J$.

In figures 2 and 3 the variations of $|R|$ and $|T|$ are indicated as functions of S for some values of Ri_{m}. Note that the values of S for which the maxima of $|R|$ and $|T|$ are reached are greater than 0.75, i.e. $Ri_{\text{c}} > Ri_{\text{m}}$.

Overreflection starts with $Ri_{\text{m}} = 0.143$, while at the critical level $Ri_{\text{c}} = 0.169$. Lower values are found for one critical level. Jones (1968) calculated $Ri = 0.113$; Eltayeb & McKenzie (1975) obtained 0.115 and van Duin & Kelder (1982) 0.132. The background mean flow considered by Jones and Eltayeb & McKenzie was formed by matching constant shear layers, which implies that the Richardson number is constant in each layer. The value given by van Duin & Kelder for the hyperbolic-tangent profile corresponds to $Ri_{\text{c}} = Ri_{\text{m}}$. The higher value found here can be explained by the fact that, as we shall show, the upper critical level acts as a source of wave energy.

In regions where $\overline{U} \neq 0$ (3.12) does not represent the total vertical energy flux. Another term which represents the advection by wave field of the kinetic energy of the mean flow is needed (Hines & Reddy 1967; Lindzen 1973), and this leads to

$$F_E = \rho_0 \overline{\phi w} + \rho_0 \overline{U}\overline{uw}. \tag{4.15}$$

In fact, what we need is a quantity whose flux is conserved across the jet except at critical levels. We could as well use other quantities such as horizontal wave momentum (Eliassen & Palm 1962) or wave action (Bretherton 1969; Andrews & McIntyre 1978; Grimshaw 1984). In the present case the flux of these quantities agrees with the flux of total energy as defined above up to a multiplicative constant.

From (3.13) and (2.1), the horizontal perturbation velocity may be written as

$$u = \frac{\overline{U}_z}{N^2}\phi_z + \frac{k}{\Omega}\phi, \tag{4.16}$$

which allows us to write the vertical energy flux as

$$F_E = \tfrac{1}{2}\rho_0 \operatorname{Re}[\mathrm{i}\sigma\phi\phi_z^*]. \tag{4.17}$$

Near the upper critical level, where $y-1 > 0$ and $Ri_{\text{c}} > \frac{1}{4}$, $\phi(y)$ can be written

$$\phi = \exp\left[\frac{Ddy}{2H}\right][P(y-1)^{\frac{1}{2}(1+\mathrm{i}\mu)} + Q(y-1)^{\frac{1}{2}(1-\mathrm{i}\mu)}], \tag{4.18}$$

where P and Q are complex constants.

This last expression used in (4.17) gives

$$F_E^+ = \frac{\sigma\mu}{4Dd}[|P|^2 - |Q|^2]. \tag{4.19}$$

The analytical continuation of ϕ for $y-1<0$ through $-\pi$ is

$$\phi = [-\mathrm{i}P\,\mathrm{e}^{\mu\pi}(y-1)^{\frac{1}{2}(1+\mathrm{i}\mu)} - \mathrm{i}Q\,\mathrm{e}^{-\mu\pi}(1-y)^{\frac{1}{2}(1-\mathrm{i}\mu)}]\exp\left[\frac{Ddy}{2H}\right], \tag{4.20}$$

which gives

$$F_E^- = \frac{\sigma\mu}{4dD}[|Q|^2\,\mathrm{e}^{-2\mu\pi} - |P|^2\,\mathrm{e}^{+2\mu\pi}]. \tag{4.21}$$

If, as expected, there is an upward-propagating wave above the upper critical level, F_E^+ is positive; then F_E^- is negative, and the upper critical level acts as a source of energy.

For $Ri_c < \frac{1}{4}$ it is not possible to prove the same statement, but the numerical calculation shows that F_E^- is always smaller than F_E^+.

We conclude that the wave is partly reflected and partly transmitted at the lower critical level; the transmitted part reaches the upper critical level, and is then reflected. The downward energy flux is added to the energy already reflected by the lower critical level, and contributes to the total wave reflection. Nevertheless, in order to observe overreflection, it is necessary that the wave absorption at the lower critical level, if any, remains very small.

We can anticipate a result obtained by numerical calculation. If the profile that gives $|R| = 1$ ($Ri_c = 0.169$, $Ri_m = 0.143$) is modified in its upper part by taking a constant mean flow above its maximum, the upper critical level is suppressed. In that case the reflection coefficient becomes $|R| = 0.91$.

The transmission coefficient depends on Ri_c and on the mean shear flow below the critical levels. If the mean shear below the critical levels is not very strong the transmission coefficient $|T| = \exp[-2\pi(Ri_c - \frac{1}{4})^{\frac{1}{2}}]$ is consistent with Booker and Bretherton's results. If the shear is strong (low value of Ri_m) even for large values of Ri_c, $|T|$ becomes lower than the product of the transmission coefficients because the wave is partially reflected below the critical level. As an example, the case $Ri_m = Ri_c = 5$ gives $|R| = 2.9\times10^{-4}$ and $|T| = 1.29\times10^{-6}$. Note that this value of $|T|$ corresponds to the product of two separated critical levels. With the same $Ri_c = 5$, but with $Ri_m = 0.12$, the results are $|R| = 0.86$ and $|T| = 3\times10^{-7}$.

Comparison of our reflection coefficients with that found by Drazin *et al.* (1979) for the triangular jet shows that our values are much lower. This can be explained by the reflection at the knees of the broken-line profile of Drazin *et al.* (see e.g. Jones 1968; Eltayeb & McKenzie 1975).

5. Numerical calculation and the asymmetric jet

Equation (3.3) has been solved numerically by the method of Bulrisch & Stoer (1966). If we introduce an imaginary component for the frequency (about 10^{-7} of the real component) the singularities are shifted off the real axis and integration along this axis is possible. We start from large positive values of y with an upward-propagating wave – the equation is integrated backwards down to a level below the jet at a distance where the solutions are again well approximated by plane waves. The solution can be uniquely decomposed into an incident and a reflected wave, and

Ri_c^u	Ri_c^l	$\|R\|$	$\|T\|$
0.12	0.12	1.32	0.70
0.07	0.07	2.11	1.63
0.12	0.07	1.74	1.00
0.07	0.12	1.47	1.00
5	5	2.64×10^{-3}	1.13×10^{-6}
2	2	3.25×10^{-2}	2.45×10^{-4}
5	2	3.26×10^{-2}	1.66×10^{-5}
2	5	3.00×10^{-3}	1.66×10^{-5}

TABLE 1. Reflection and transmission coefficients for asymmetric jets with $S = 0.86$

so R and T can be determined. The integration step is changed automatically when the desired accuracy is not found. For example, the minimum step near the critical level can be as low as 10^{-10} of the total integration path with no more than 500 steps.

The differences between the numerical and the analytical results are less than 1 %.

An asymmetric jet can be defined as

$$U(z) = \begin{cases} \dfrac{U_0}{1+z^2/D_1^2} & (z < 0), \\ \dfrac{U_0}{1+z^2/D_2^2} & (z > 0). \end{cases}$$

These profiles were also considered numerically.

In table 1 some calculated values are listed. The values of the Richardson number at the lower critical level and at upper critical level of the jet, Ri_c^l and Ri_c^u respectively, have been used as parameters. The calculation was performed with $S = 0.86$.

Two conclusions can be drawn from table 1:

(*a*) changes in Ri_c^u modify the reflection coefficient – this proves that part of the reflected wave comes from the upper critical level;

(*b*) interchange of Ri_c^l and Ri_c^u changes $|R|$ but leaves $|T|$ invariant.

Both results are consistent with the hypothesis of a wave partially trapped between the two critical levels.

6. One critical level

When the horizontal phase velocity of the wave equals the maximum of the mean-flow velocity there is only one critical level. Mathematically, the singularity of (2.5) becomes an irregular singularity. However, when a small dissipation term is added the irregular singularity splits into two regular ones. Thus the case of one critical level has been reduced to the case of two merging critical levels.

Let us start with two critical levels and let $c \to U_0$, i.e. $S \to 1$. From (4.14) it is clear that $\gamma^2 \to 0$, $\mu \to \mathrm{i}\infty$ and $\lambda \to -2J$. We calculated the values of R and T in this limit.

From Stirling's formula for gamma functions it follows that

$$\Gamma(\mu+\nu+1) \approx \mathrm{e}^{-\mu}\mu^{\mu+\nu+\frac{1}{2}}(2\pi)^2 \tag{6.1}$$

and

$$\Gamma(\mu-\nu) \approx \mathrm{e}^{-\mu}\mu^{\mu-\nu-\frac{1}{2}}(2\pi)^2 \tag{6.2}$$

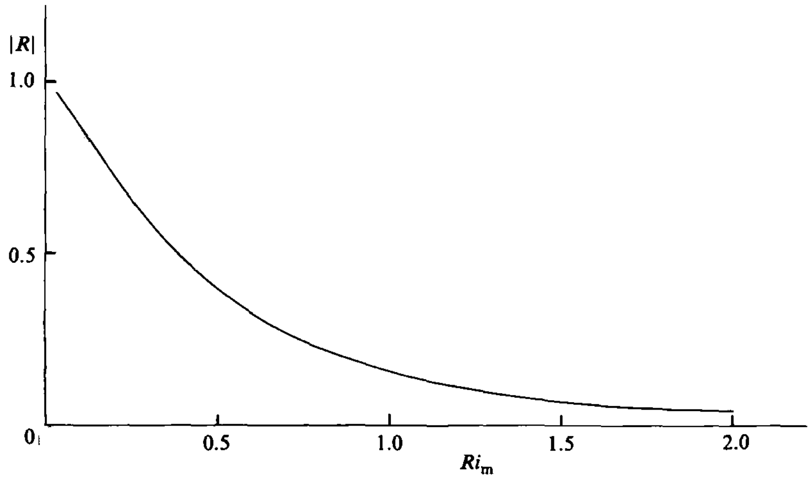

FIGURE 4. Variation of the reflection coefficient as a function of the minimum Richardson number of the mean flow in the case of two merging critical levels.

for $\mu \rightarrow \mathrm{i}\infty$. From (4.2) it may be inferred that for $\gamma \rightarrow 0$

$$\frac{V_\nu^\mu}{V_{-\nu-1}^\mu} \approx \left(\frac{\gamma}{4}\right)^{2\nu+1} \left(\frac{\Gamma(-\nu+\frac{1}{2})}{\Gamma(\nu+\frac{3}{2})}\right)^2. \tag{6.3}$$

Moreover, if we put $\mu = \mathrm{i}M$ then

$$\sin(\nu+\mu)\pi \approx \tfrac{1}{2}\mathrm{i}\, \mathrm{e}^{(M-\mathrm{i}\nu)\pi}, \tag{6.4}$$

$$\sin(\nu-\mu)\pi \approx -\tfrac{1}{2}\mathrm{i}\, \mathrm{e}^{(M+\mathrm{i}\nu)\pi}, \tag{6.5}$$

$$\cos \mu\pi \approx \tfrac{1}{2}\mathrm{e}^{M\pi}. \tag{6.6}$$

From (6.1)–(6.6) it can be derived that

$$V^+ \approx \tfrac{1}{2}\mathrm{i}\, \mathrm{e}^{(M-\mathrm{i}\nu)\pi} \left(\frac{\gamma\mu}{4}\right)^{2\nu+1} \left(\frac{\Gamma(-\nu+\frac{1}{2})}{\Gamma(\nu+\frac{3}{2})}\right)^2. \tag{6.7}$$

Taking into account that $\mu \approx \mathrm{i}\, ND/U_0 d$ and $\gamma \rightarrow Dld$, we can write

$$V^+ \approx -\tfrac{1}{2}\, \mathrm{e}^{M\pi} F, \tag{6.8}$$

$$V^- \approx -\frac{\mathrm{e}^{M\pi}}{2F}, \tag{6.9}$$

where

$$F = \left(\frac{ND^2 l}{4U_0}\right)^{2\nu+1} \left(\frac{\Gamma(-\nu+\frac{1}{2})}{\Gamma(\nu+\frac{3}{2})}\right)^2. \tag{6.10}$$

Introducing (6.8) and (6.9) in (4.8) and (4.9) to calculate p and q, and using these results in (4.12) and (4.13), we obtain for the reflection and transmission coefficients

$$R = \frac{\sin\nu\pi + \frac{1}{2}\mathrm{i}\,[F^{-1}\,\mathrm{e}^{\mathrm{i}\nu\pi} - F\,\mathrm{e}^{-\mathrm{i}\nu\pi}]}{\mathrm{i}\,[1+\frac{1}{2}(F^{-1}\,\mathrm{e}^{2\mathrm{i}\nu\pi} + F\,\mathrm{e}^{-2\mathrm{i}\nu\pi})]}, \tag{6.11}$$

$$T = \frac{\mathrm{e}^{-M\pi}\, 2\cos^2\nu\pi}{1+\frac{1}{2}(F^{-1}\,\mathrm{e}^{2\mathrm{i}\nu\pi} + F\,\mathrm{e}^{-2\mathrm{i}\nu\pi})}. \tag{6.12}$$

In the limit $S \to 1$, that is $\gamma \to 0$, we can deduce from (3.7) that

$$\nu \to -\tfrac{1}{2}-\tfrac{1}{2}(1+4\lambda)^{\frac{1}{2}} \approx -\tfrac{1}{2}-\tfrac{1}{2}(1-8J)^{\frac{1}{2}}, \tag{6.13}$$

which shows that ν takes on a finite value in this limit.

From (6.12) it is clear that as $S \to 1$ the transmission coefficient $T \to 0$. The wave is not transmitted. This result is consistent with Booker & Bretherton's result, as, for $S \to 1$, $Ri_c \to \infty$.

Numerical calculation with (6.11) shows that $|R|^2 \leqslant 1$.

A wave propagating through a jet with two merging critical levels will not be overreflected, but only partially reflected and not transmitted.

Figure 4 shows that $|R|$ decreases with increasing values of Ri_m.

7. Conclusion

This work was motivated by the frequent observations of jet-type background winds in the atmosphere and the fact that planetary waves are seen as stationary jets by short-period gravity waves.

In such a background flow a gravity wave can have two critical levels or only one with specific characteristics. In this study we have shown that the two critical levels do not act independently to determine the behaviour of the travelling wave. In fact, some of the energy transmitted through the lower critical level can be reflected at the upper one, which acts as a source of wave energy. Then the downward energy flux is added to the energy already reflected at the lower critical level and can produce overreflection, with Ri_c higher than in the case of only one critical level. We found overreflection with $Ri_c = 0.169$, the critical level located at $S = 0.86$. On the other hand, even for values of Ri_c as low as 0.1, and with $Ri_c = Ri_m$ ($S = 0.75$), neither overreflection nor overtransmission occurs.

The transmission coefficient is different from the product of the transmission coefficients of the two critical levels when $0.25 \leqslant Ri_m < 1$ because the wave is partially reflected below the critical levels by the strong shear.

The results found for asymmetric jets show that the upper critical level can affect the reflection of the wave.

The case of a wave having only one critical level at the maximum of the background flow has been solved as the limit of two merging critical levels. In this case the transmission becomes zero and no overreflection can occur.

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